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14. ABSTRACT

IN this talk ppt given at Dr Mubarak Shah's Center for Research in Vision at U Central Florida, Orlando, March 25 - 26 2013, we describe the overarching principles that guides the recent work on Tipping points and committed Minorities in SIgnalling Multi-Agent Social Networks

We find that a scalar stochastic Differential equation can be derived to give good estimates of the first exit times

15. SUBJECT TERMS

Tipping Point, Times to consensus, scalar SDE

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Report Title

Organizing Principle for Tipping Points in Social Networks

ABSTRACT

IN this talk ppt given at Dr Mubarak Shah's Center for Research in Vision at U Central Florida, Orlando, March 25 - 26 2013, we describe the overarching principles that guides the recent work on Tipping points and committed Minorities in SIgnalling Multi-Agent Social Networks

We find that a scalar stochastic Differential equation can be derived to give good estimates of the first exit times such as consensus times and to study the influence of diehards.

For very large populations we find that a one-dimensional center manifold on which the mean-field dynamics in slow time is easily shown to consist of nodes and saddle and heteroclinic orbits linking these equilibria - the saddle node bifurcation then led to an easy determination of the critical value of diehard fraction needed to accelerate the network to consensus in the minority opinion.

For smaller crowds on the order of 1000 - 5000 agents, we find that the demographic noise reflected in simulations by the significant variance in consensus times, must be taken into account to determine accurate estimates of the expected value of consensus times and to study the effects of committed minority.

Organizing Principles in Network Science- Scalar SDE,Tipping Points of opinion dynamics

Chjan Lim, Mathematical Sciences, RPI

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- Secondary: ARL grants 2009 – 2012

Misc

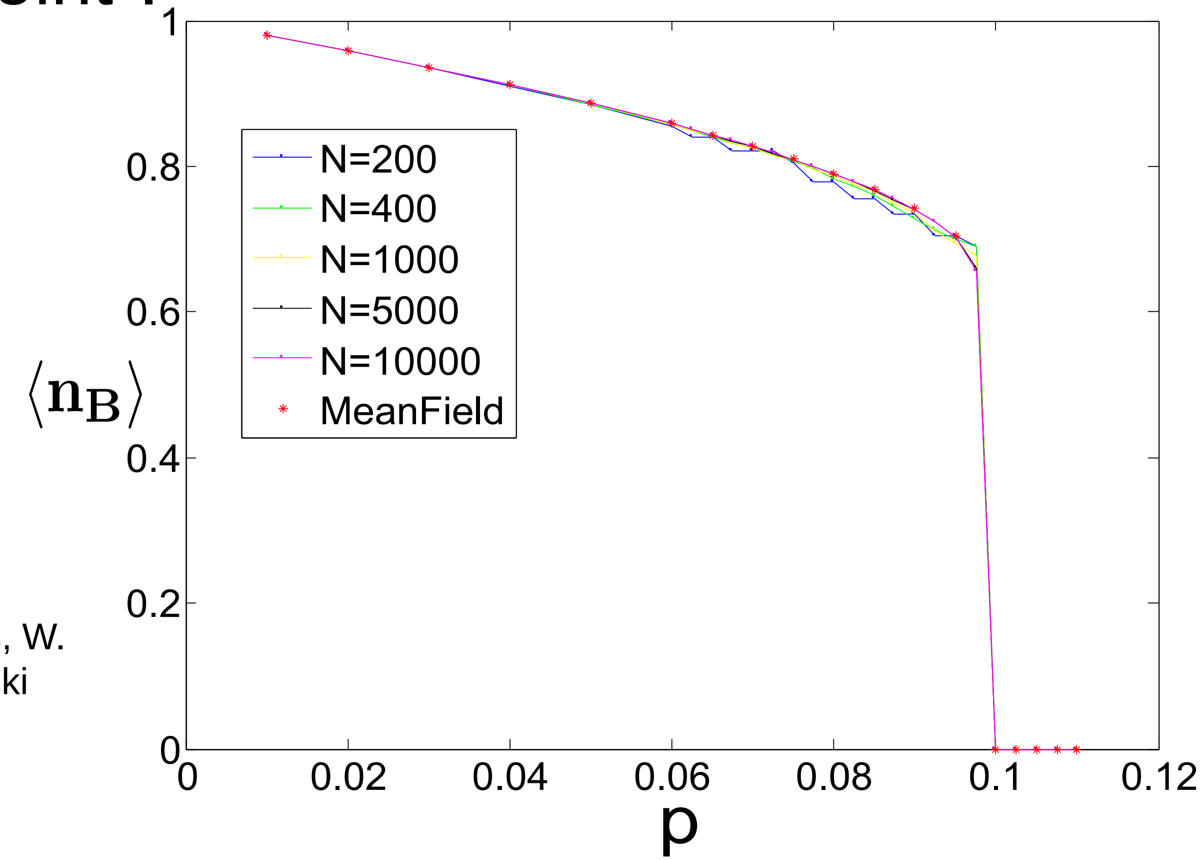
- Collaborators: Dr W Zhang, **Y Treitman**, B Szymanski, G Korniss
- Papers:
- Weituo Zhang, Chjan C. Lim, B. Szymanksi, “Analytic Treatment of Tipping Points for Social Consensus in Large Random Networks”, Phys Rev E 86 (6), 061134, 2012

  Weituo Zhang, Chjan Lim, and Boleslaw Szymanski, Tipping Points of Diehards in Social Consensus on Large Random Networks, Complex Networks, Proc. 3rd Workshop on Complex Networks, CompleNet, Melbourne, FL, March 7-9, 2012, Studies in Computational Intelligence, vol. 424, Springer, Berlin, Germany, 2013, pp. 161-168.

Yosef Treitman, Chjan Lim, W. Zhang and A. Thompson, “Naming Game with Greater Stubbornness and Unilateral Zealots”, IEEE NSW Conf., April 29 – May 1 2013, West Point, NY

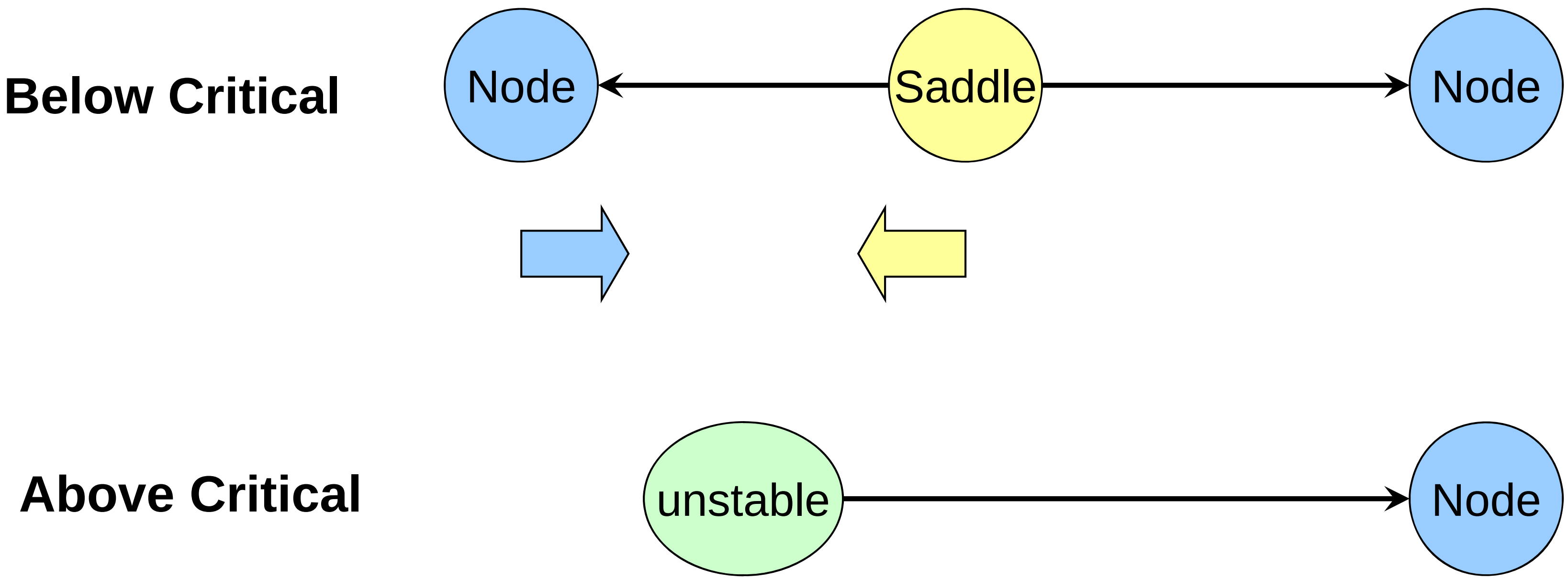
Tipping Point of NG

- A minority of committed agents can persuade the whole network to a global consensus.
- The critical value for phase transition is called the “tipping point”.



J. Xie, S. Sreenivasan, G. Korniss, W. Zhang, C. Lim and B. K. Szymanski PHYSICAL REVIEW E (2011)

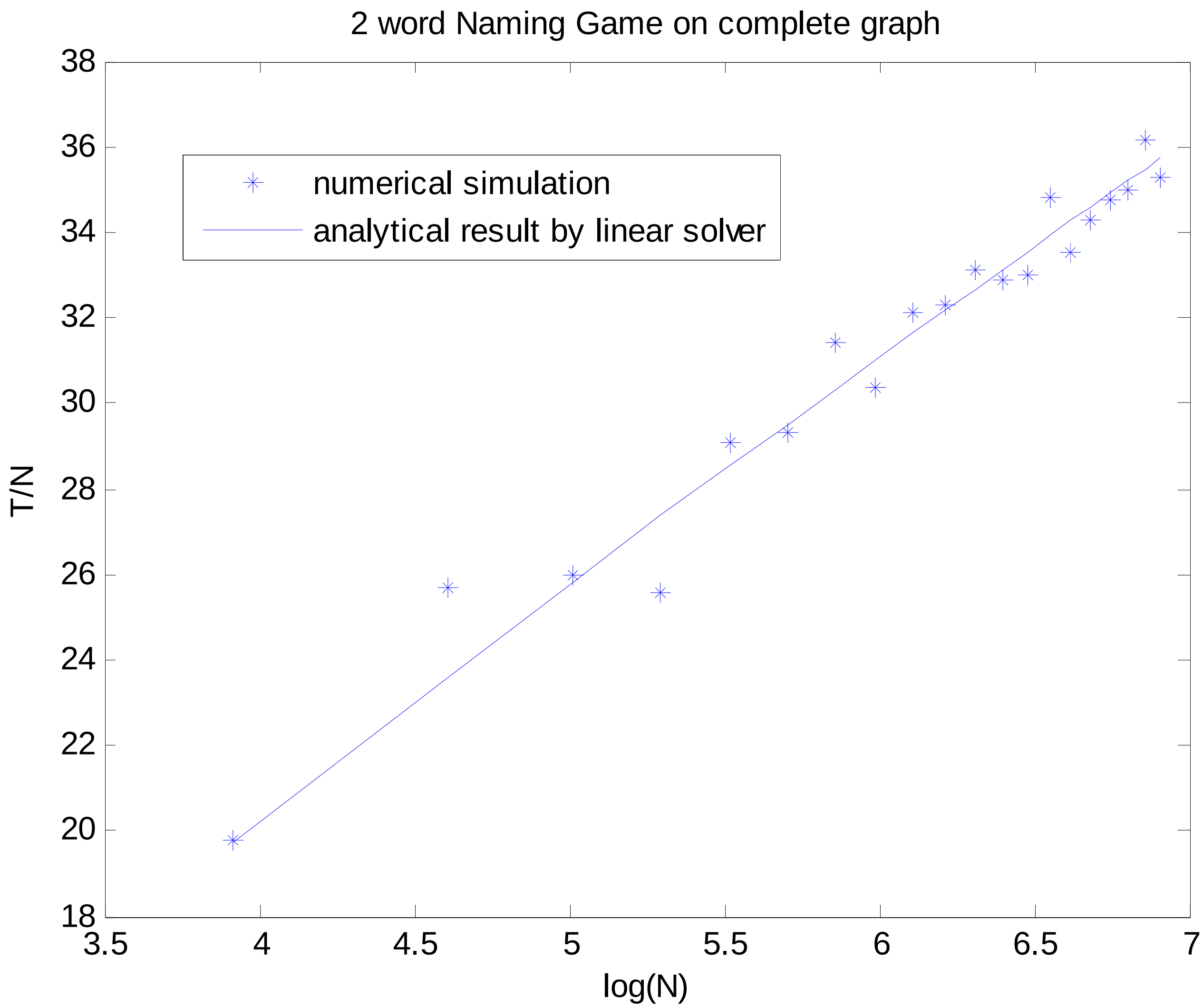
Saddle node bifurcation



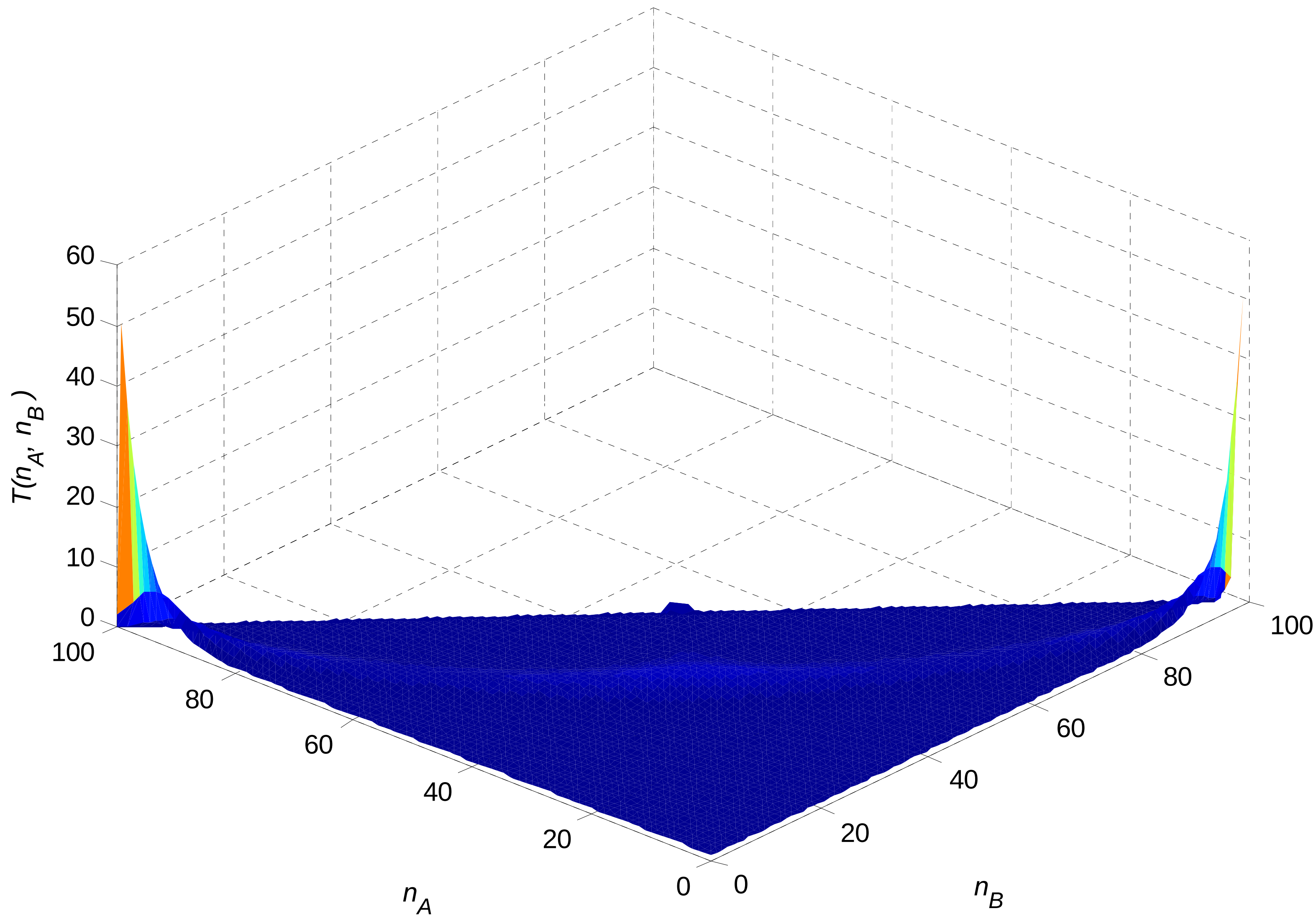
Meanfield Assumption and Complete Network

- The network structure is ignored. Every node is only affected by the meanfield.
- The meanfield depends only on the fractions(or numbers) of all types of nodes.
- Describe the dynamics by an equation of the meanfield (macrostate).

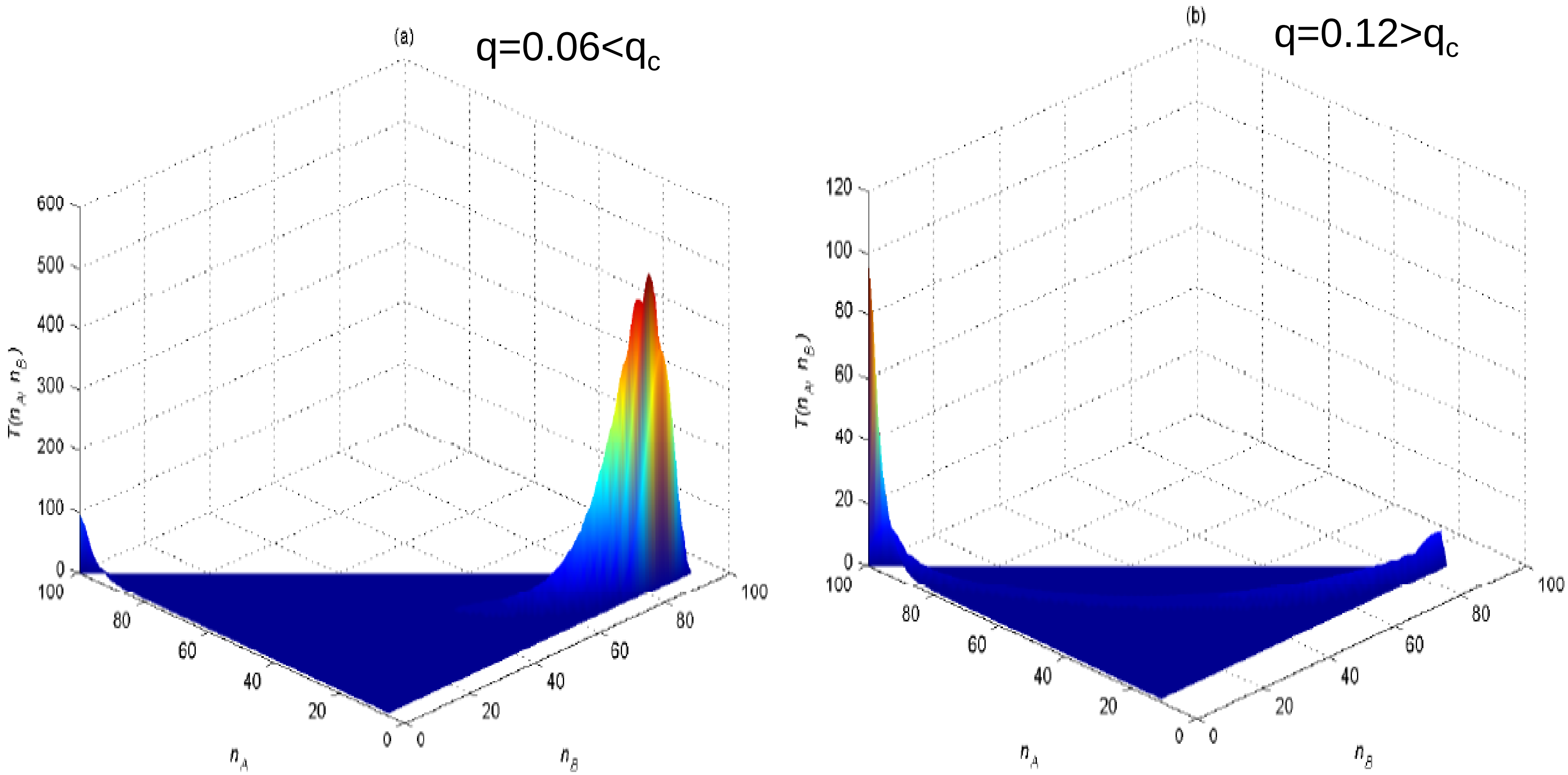
Scale of consensus time on complet graph



Expected Time Spend on Each Macrostate before Consensus (without committed agents)



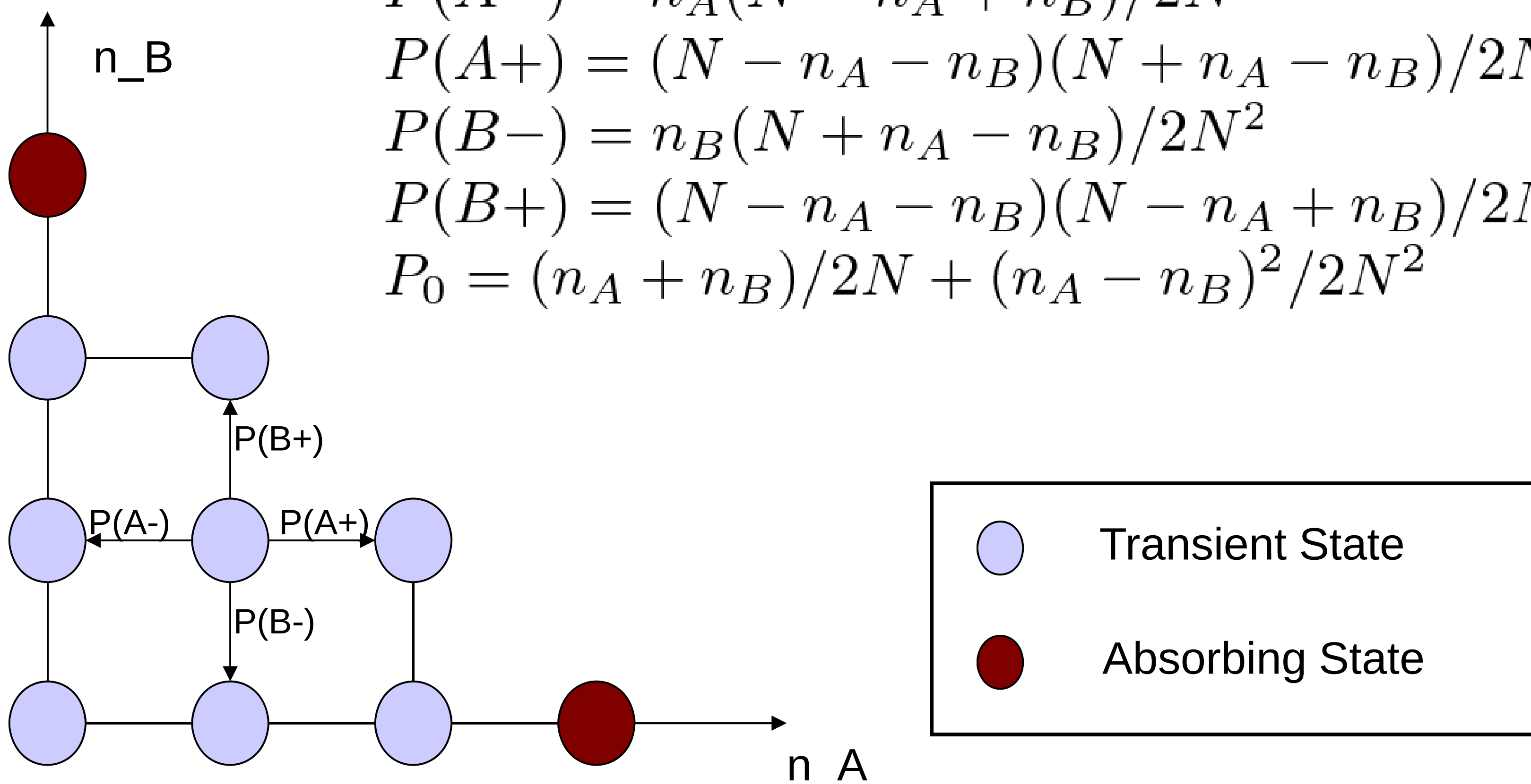
NG with Committed Agents



q is the fraction of agents committed in A. When q is below a critical value q_c, the process may stuck in a meta-stable state for a very long time.

2 Word Naming Game as a 2D random walk

$$P(A-) = n_A(N - n_A + n_B)/2N^2$$
$$P(A+) = (N - n_A - n_B)(N + n_A - n_B)/2N^2$$
$$P(B-) = n_B(N + n_A - n_B)/2N^2$$
$$P(B+) = (N - n_A - n_B)(N - n_A + n_B)/2N^2$$
$$P_0 = (n_A + n_B)/2N + (n_A - n_B)^2/2N^2$$



Linear Solver for 2-Name NG

$\tau(n_A, n_B)$ is the absorbing time starting from network state (n_A, n_B).
$t(n_A, n_B)$ is the average time stay in network state (n_A, n_B) before leaving.

Have equations:

$$\tau(n_A, n_B) = \frac{P(A+)\tau(n_A+1, n_B) + P(A-)\tau(n_A-1, n_B) + P(B+)\tau(n_A, n_B+1) + P(B-)\tau(n_A, n_B-1)}{1-P_0} + t(n_A, n_B)$$

Then we assign an order to the coordinates, make $\tau(n_A, n_B)$, $t(n_A, n_B)$ into vectors, and finally write equations in the linear system form:

$$\vec{\tau} = M\vec{\tau} + \vec{t}$$

SDE models for NG, NG and NG

$$d\vec{X} = \vec{\mu}dt + \frac{\sigma}{\sqrt{N}}d\vec{W}$$

Diffusion vs Drift

- Diffusion scales are clear from broadening of trajectories bundles
- Drift governed by mean field nonlinear ODEs can be seen from the average / midlines of bundles

Assume a very natural social – political condition, generalizing NG

Where the network is divided into $k+1$ sub-populations

Each with a different fixed propensity to signal/vote/utter the opinion A

When hearing the word A a node from subgroup $j < k$ will move to subgroup $j+1$
Likewise hearing B a node from $j > 0$ will move to subgroup $j-1$

The probability that a node from subgroup j will signal A is j/k

Same node has probability $1 – j/k$ of signalling word B

Additional rules of k-NG

- A node s is drawn at random from the network and sends out a signal A with p
- or s signals B with prob = 1 – p
- Next a node L is drawn at random to receive the signal A
- or node L is drawn to receive the signal B

k-NG

- The subgroups $j = 0, k$ correspond to those nodes that are completely convinced of the B and A opinion resp. OR

- Equivalently those nodes that with prob 1
- Signals B, A resp.

Voting or Polling

- The key network quantity is an average network opinion obtained by polling:
-
- p = sum over subgrps j = 0 to k
- of n(j) j / kN
- It gives the probability of a speaker chosen at random signalling the word A

Stochastic Dynamics

- Derivation of k-dim coupled Random Walk:

- For j = 1 to k-1,
- n(j,t+1) = n(j,t) + 1, -1 with resp. prob.
- P(+1) = p(t) n(j-1,t)/N + (1-p(t)) n(j+1,t)/N
- P(-1) = (1-p(t)) n(j,t)/N + p(t) n(j,t)/N
- = n(j,t)/N

Random walk

- For the distinguished subgrps at both ends of the opinion spectrum, $j = 0$, k

- $n(0,t+1) = n(0,t) + 1, 0, -1$ with prob.
- $P(+1) = (1-p(t))\, n(1,t)/N$
- $P(0) = (1-p(t))\, n(0,t)/N$
- $P(-1) = p(t)\, n(0,t)/N$

Subgroup k

- $n(k,t+1) = n(k,t) + 1, 0, -1$ with prob.

- $P(+1) = p(t)\, n(k-1,t)/N$
- $P(0) = p(t)\, n(k,t)/N$
- $P(-1) = (1-p(t))\, n(k,t)/N$

Shadow Walk

- Define a scalar nonlazy random walk on the quantity p whose consensus times are lower bounds for the actual expected times.

- $p(t+1) = p(t) + 1/kN, \; -1/kN$ with prob

- $P(+1) = p(t)$
- $P(-1) = 1-p(t)$

SDE or Diffusion Model

In continuous time the SDE is

$$
\begin{aligned}
dP_t &= M(P_t)dt + V(P_t)dW_t \\
M &= (2P_t - 1)/k \\
V &= \frac{2}{k\sqrt{N}}\sqrt{P_t(1-P_t)}
\end{aligned}
$$

Solution of SDE

now done using standard Kolmogorov Backwards Equation method

$$
\begin{aligned}
\frac{\partial u(p,t)}{\partial t} &= \frac{V}{2}\frac{\partial^2 u}{\partial p^2} + M\frac{\partial u}{\partial p} \\
u(0,t) &= 0, u(1,t) = 1 where u(p,t) = \Pr\{P_t = 1 | P_0 = p\} \\
u(p) &= \lim_{t\to\infty} u(p,t)
\end{aligned}
$$

Expected Times to Consensus

First calculate

$$\begin{aligned} G(p) &= \exp\{-2\int \frac{M}{V} dp\} = \exp\left(2\sqrt{N}\int d\sqrt{(p-p^2)}\right) \\ &= e^{2\sqrt{N}\sqrt{p-p^{22}}}. \end{aligned}$$

Next we get

$$0 \le u(p) = \frac{\int_0^p G(x)dx}{\int_0^1 G(x)dx} = \frac{\int_0^p \exp\left(2\sqrt{N}\sqrt{x-x^2}\right)dx}{\int_0^1 \exp\left(2\sqrt{N}\sqrt{x-x^2}\right)dx} \le 1$$

Expected times - exit times – stop times

and the expected time to consensus of the A opinion without committed agents and conditioned on eventual fixation of the A opinion is given by

$$t(p) = \frac{T(p)}{u(p)}$$

where

$$T(p) = \int_0^\infty t \frac{\partial u(p,t)}{\partial t} dt$$

is the unconditioned expected time to consensus given that $P_0 = p$ and solves the stationary form of the KBE

$$\frac{d^2T}{dp^2} + \frac{2M}{V}\frac{dT}{dp} + \frac{2u(p)}{V} = 0$$

with boundary conditions

$$\begin{aligned} \lim_{p \to 0} & \quad t < \infty \\ t(1) &= 0. \end{aligned}$$

continued

The solution is given by

$$T(p) = u(p) \int_p^1 F(y)u(y)\,(1 - u(y))\,dy + (1 - u(p)) \int_0^p F(y)u^2(y)dy$$

which evaluates to the following closed form expression for the $k - NG$ problem,

$$T(p) = \frac{k\sqrt{N}}{\left[\int_0^1 G(x)dx\right]^2} \left\{ \begin{array}{c} \int_0^p G(x)dx \int_p^1 dy \frac{\int_0^y G(x)dx \int_y^1 G(x)dx}{G(y)\sqrt{y-y^2}} \\ + \int_p^1 G(x)dx \int_0^p dy \frac{\int_0^y G(x)dx \int_0^y G(x)dx}{G(y)\sqrt{y-y^2}} \end{array} \right\}$$

Higher stubbornness – same qualitative, robust result

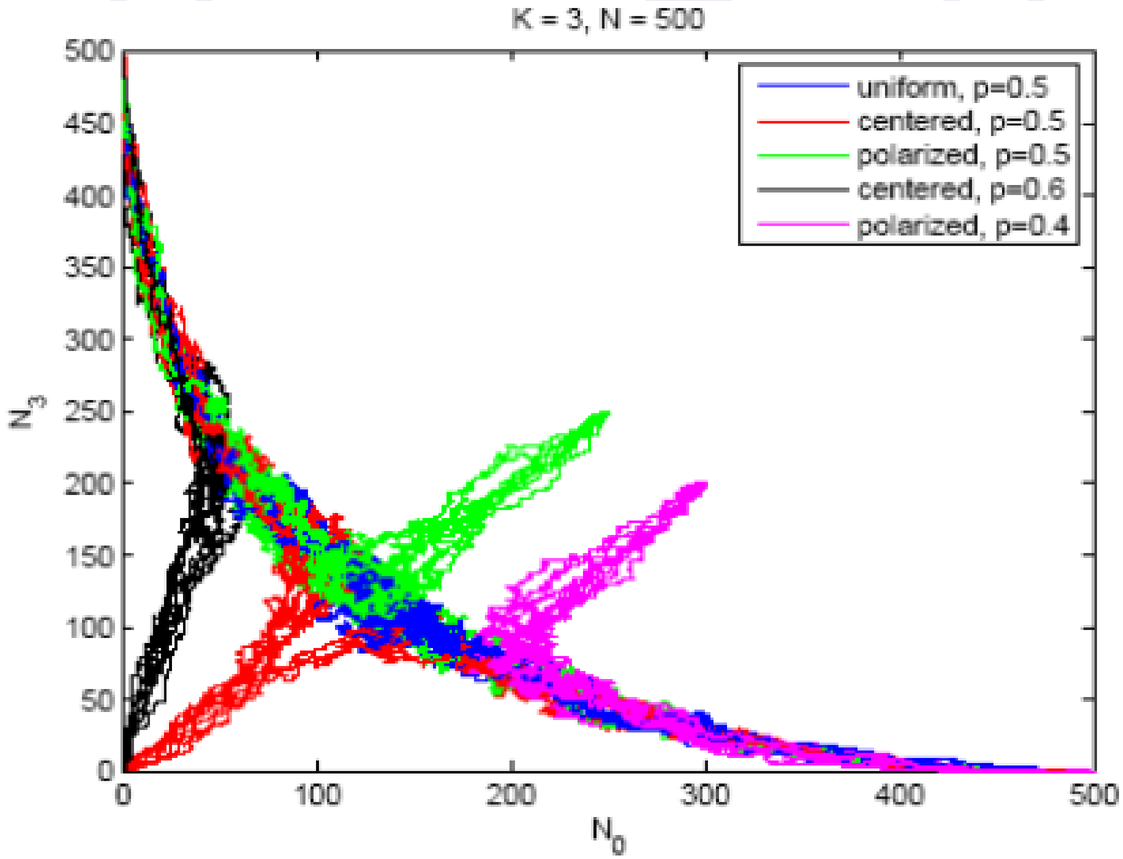


Figure 1: Simulations of the Naming Game with K=3 and N = 500. Note the existence of a center manifold to which all trajectories tend. The simulations that start far away from the center manifold approach it before drifting to a consensus state.

Higher stubbornness – same qualitative, robust result

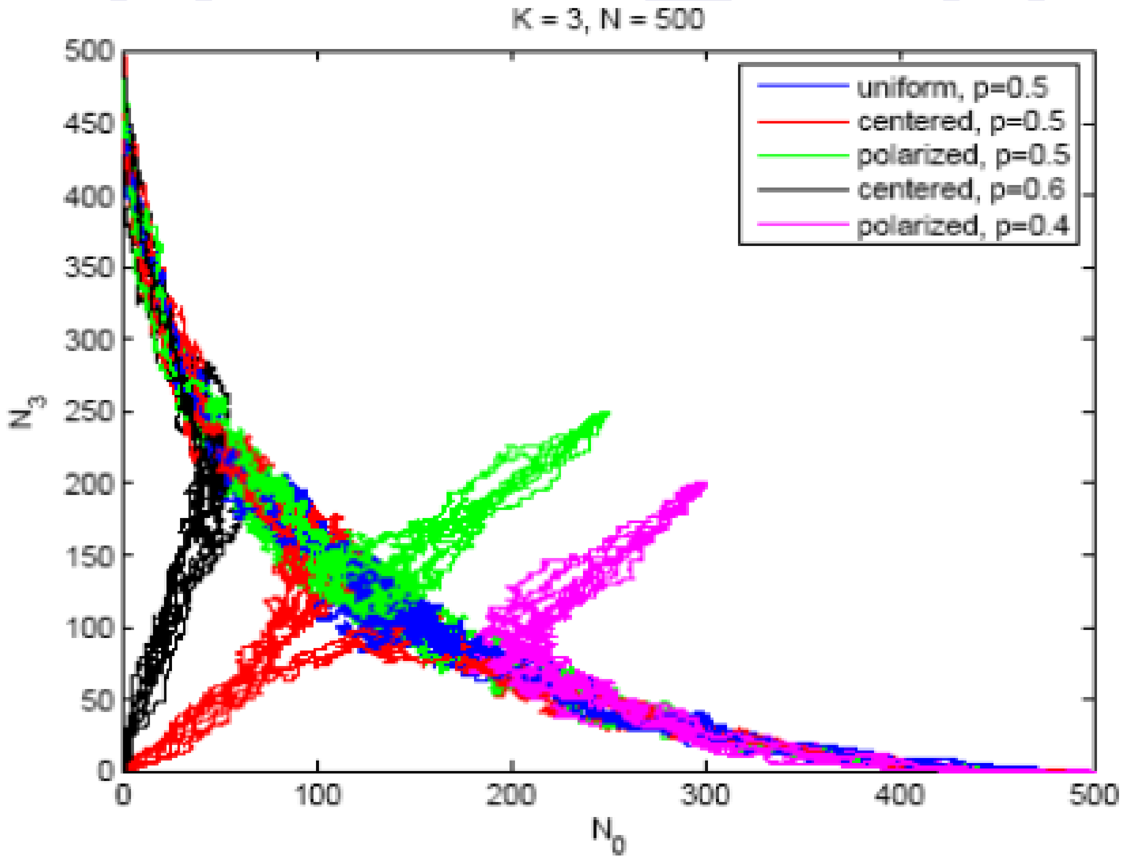


Figure 1: Simulations of the Naming Game with K=3 and N = 500. Note the existence of a center manifold to which all trajectories tend. The simulations that start far away from the center manifold approach it before drifting to a consensus state.

Other NG variants – same 1D manifold

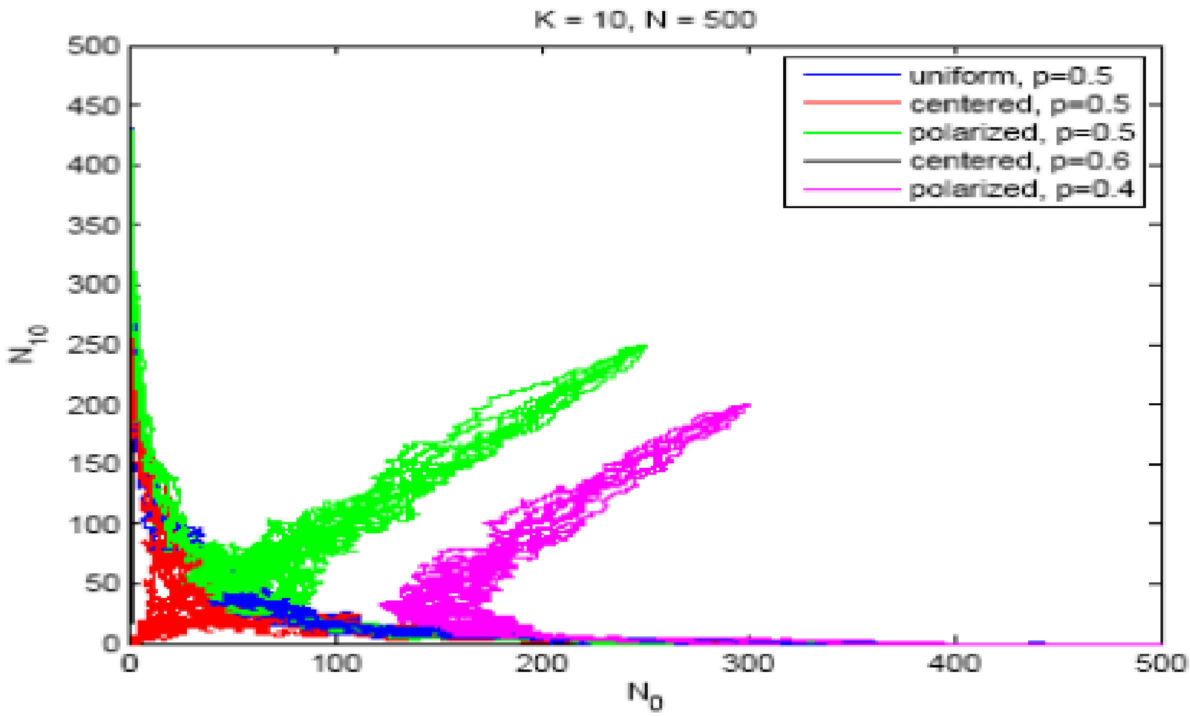


Figure 2: Simulations of the Naming Game with K=10 and N = 500. Some trajectories do not approach the center manifold as quickly as they did in the K=3 case. Those simulations that start with p=0.5 approach the center manifold before drifting to the consensus state. However, the simulations that start with a less centralized p value drift to the consensus state more quickly. Note that the simulations starting in the polarized state with p=0.4 come close to the center manifold before drifting to consensus. On the other hand, the simulations in the centralized state with p=0.6 drift to consensus before any significant number of agents enter N_0.

3D plot of trajectory bundles – stubbornness K = 10 as example of variant (Y. Treitman and C. Lim 2012)

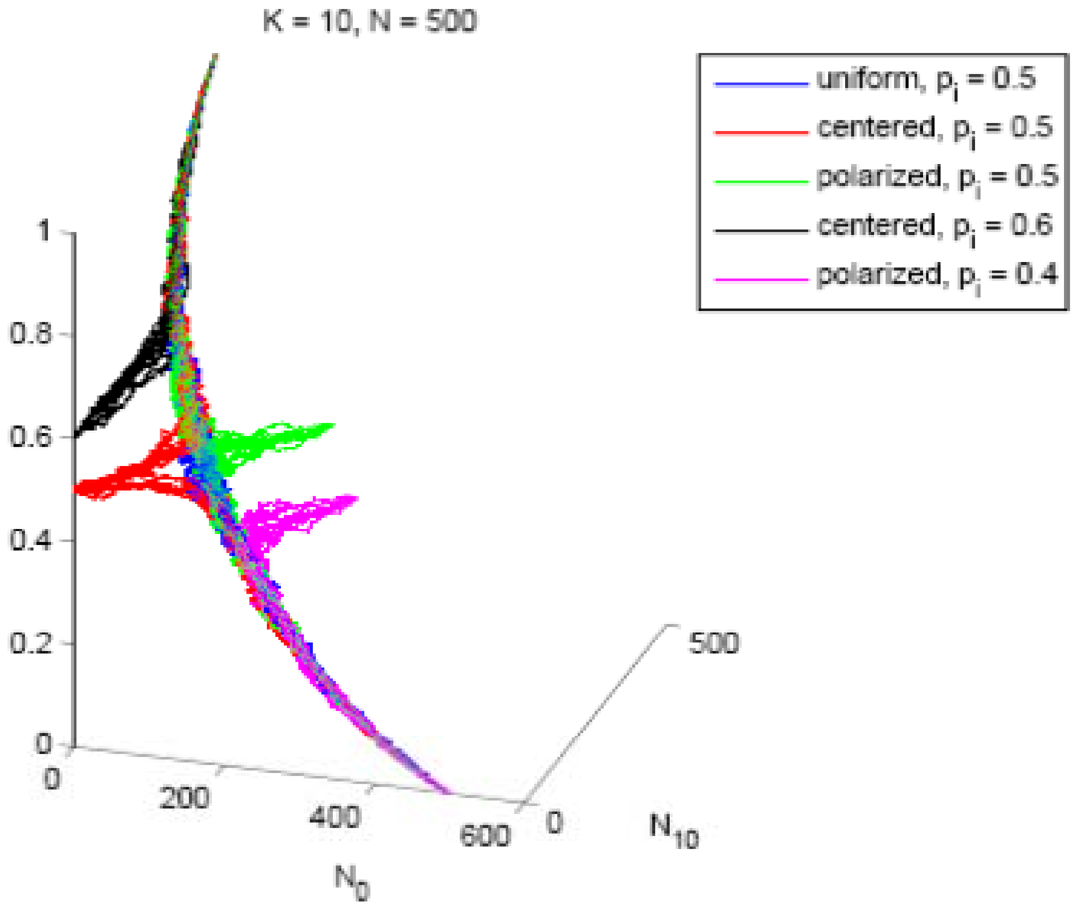


Figure 4: A 3D plot of the simulations of the Naming Game with K=3 and N = 500. Here, all simulations approach the center manifold before p deviates too much from its initial value.

Consensus time distribution

- Recursive relationship of P(X, T), the probability for consensus at T starting from X, Q is the transition matrix.

$$
\begin{aligned}
P(n_A, n_B, T+1) \;=\; & Q(n_A+1, n_B|n_A, n_B)P(n_A+1, n_B, T) + Q(n_A-1, n_B|n_A, n_B)P(n_A-1, n_B, T) \\
& + Q(n_A, n_B+1|n_A, n_B)P(n_A, n_B+1, T) + Q(n_A, n_B-1|n_A, n_B)P(n_A, n_B-1, T) \\
& + Q(n_A, n_B+2|n_A, n_B)P(n_A, n_B+2, T) + Q(n_A+2, n_B|n_A, n_B)P(n_A+2, n_B, T)
\end{aligned}
$$

Take each column for the same T as a vector:

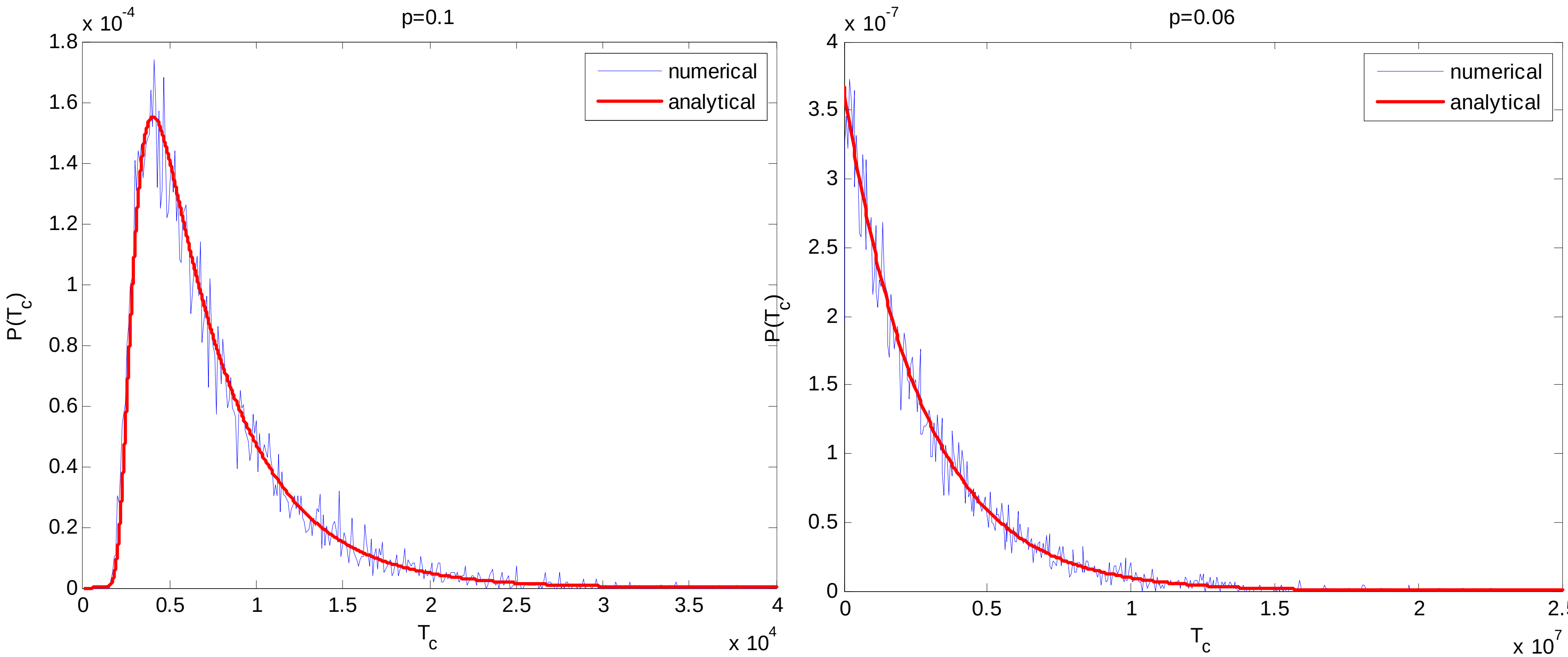
$$\vec{P}(T+1) = Q * \vec{P}(T)$$

Calculate the whole table P(X,T) iteratively.

Take each row for the same X as a vector:

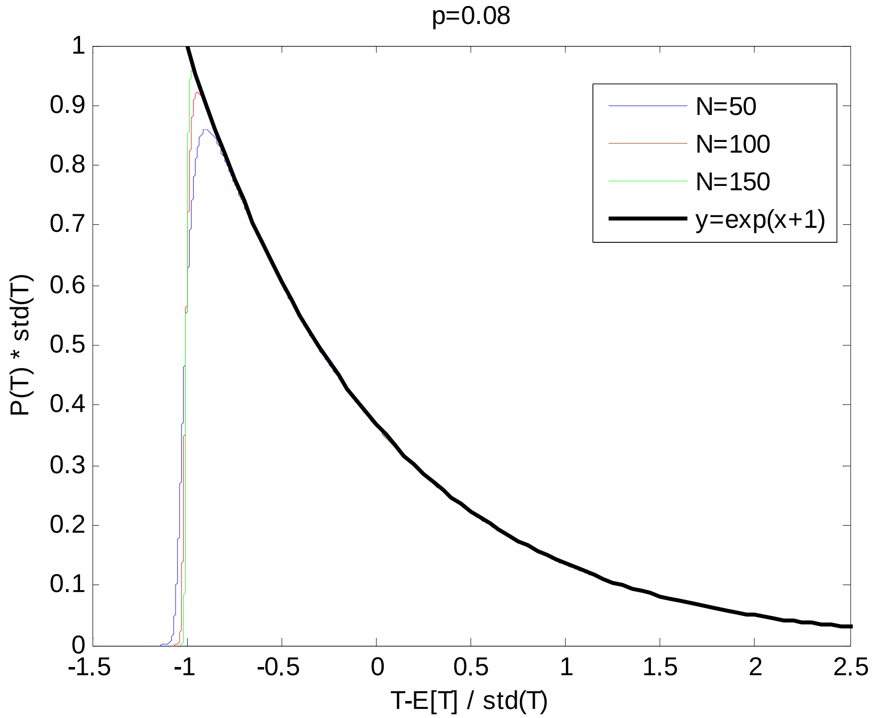
$$\mathscr{P}_{(n_A, n_B)}(T_c = T) = (P(n_A, n_B, T)).$$

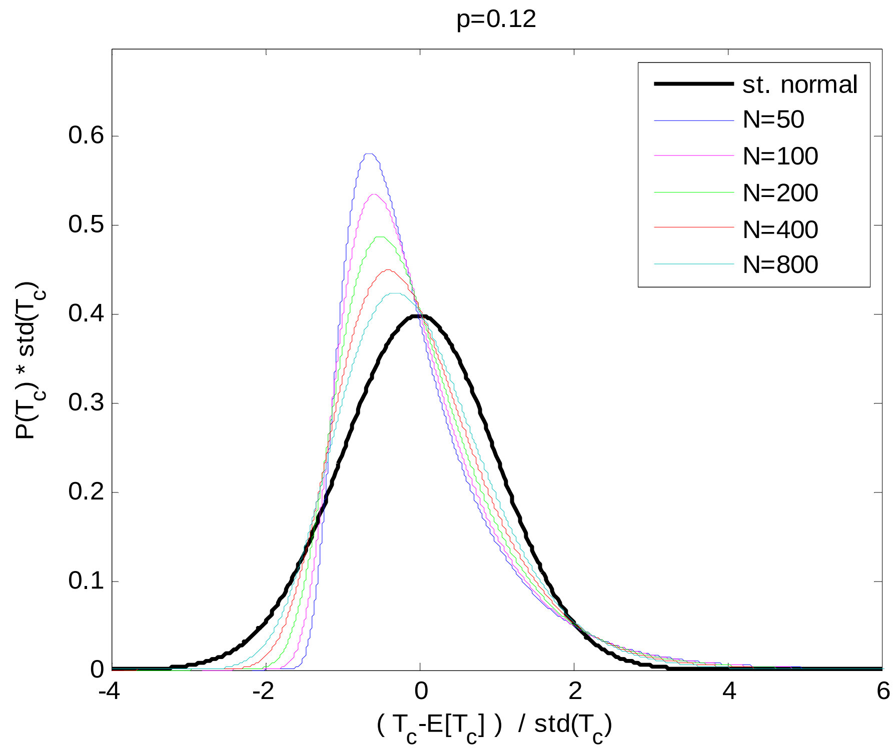
Consensus time distribution



Red lines are calculated through the recursive equation.
Blue lines are statistics of consensus times from numerical simulation(very expensive), (done by Jerry Xie)

Consensus Time distribution

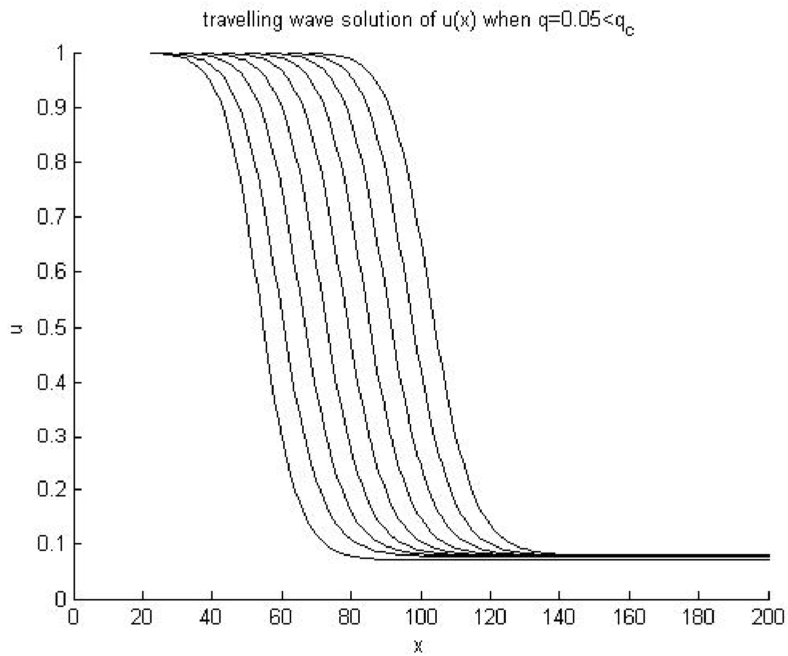




Below critical, consensus time distribution tends to exponential.
Above critical, consensus time distribution tends to Gaussian.

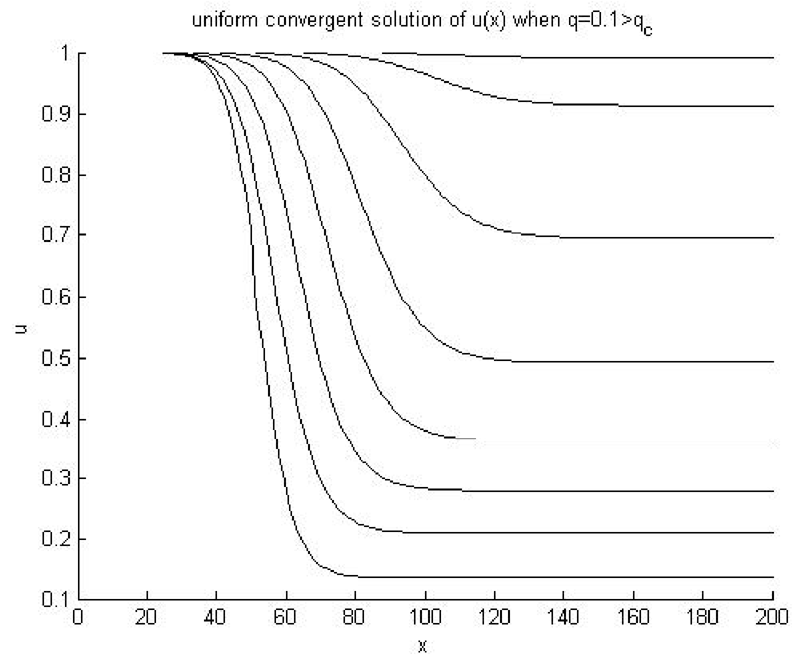
For large enough system, only the mean and the variance of the consensus time is needed.

NG on RGG



travelling wave solution of u(x) when q=0.05<q_c
u
x
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
20
40
60
80
100
120
140
160
180
200

NG on RGG past Tipping point



Homogeneous Pairwis Assumption

The mean field is not uniform but varies for the nodes with different opinion.

Make it rigorous:

suppose three nodes are linked as 1-2-3

X_i is the opinion of node i

k_i is the degree of node i

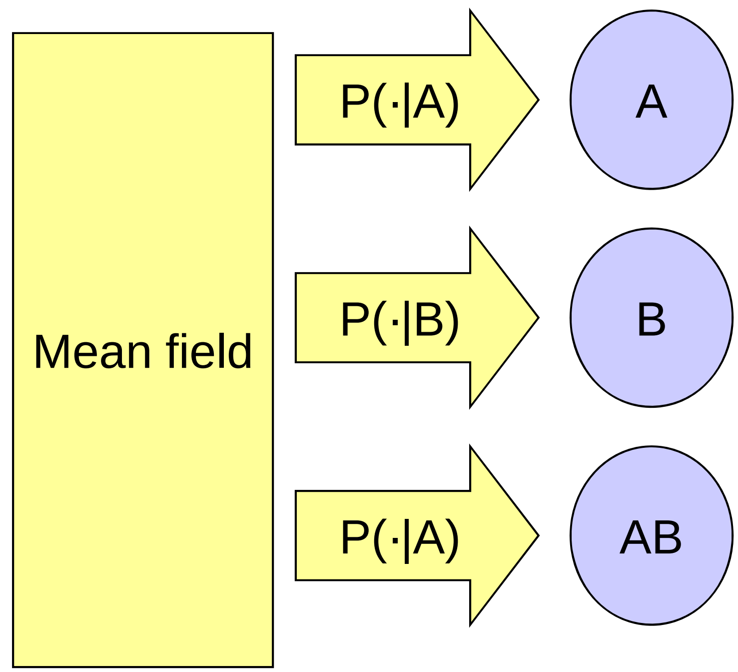
$P(X_1|X_2) \neq P(X_1)$

$P(X_1|X_2, X_3) = P(X_1|X_2)$

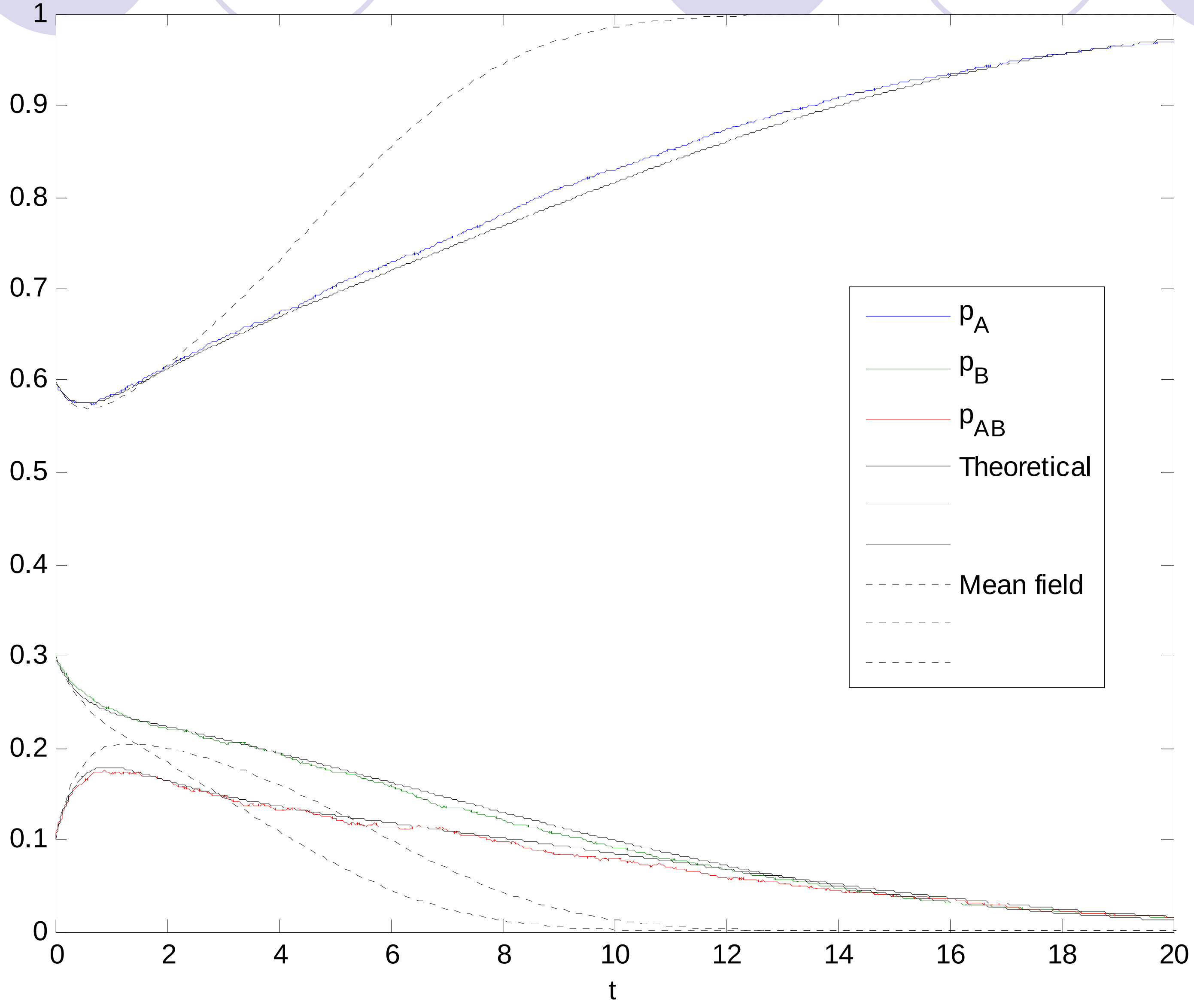
$E[k_1|X_1] = < k >$

$P(X_1|k_1) = P(X_1)$

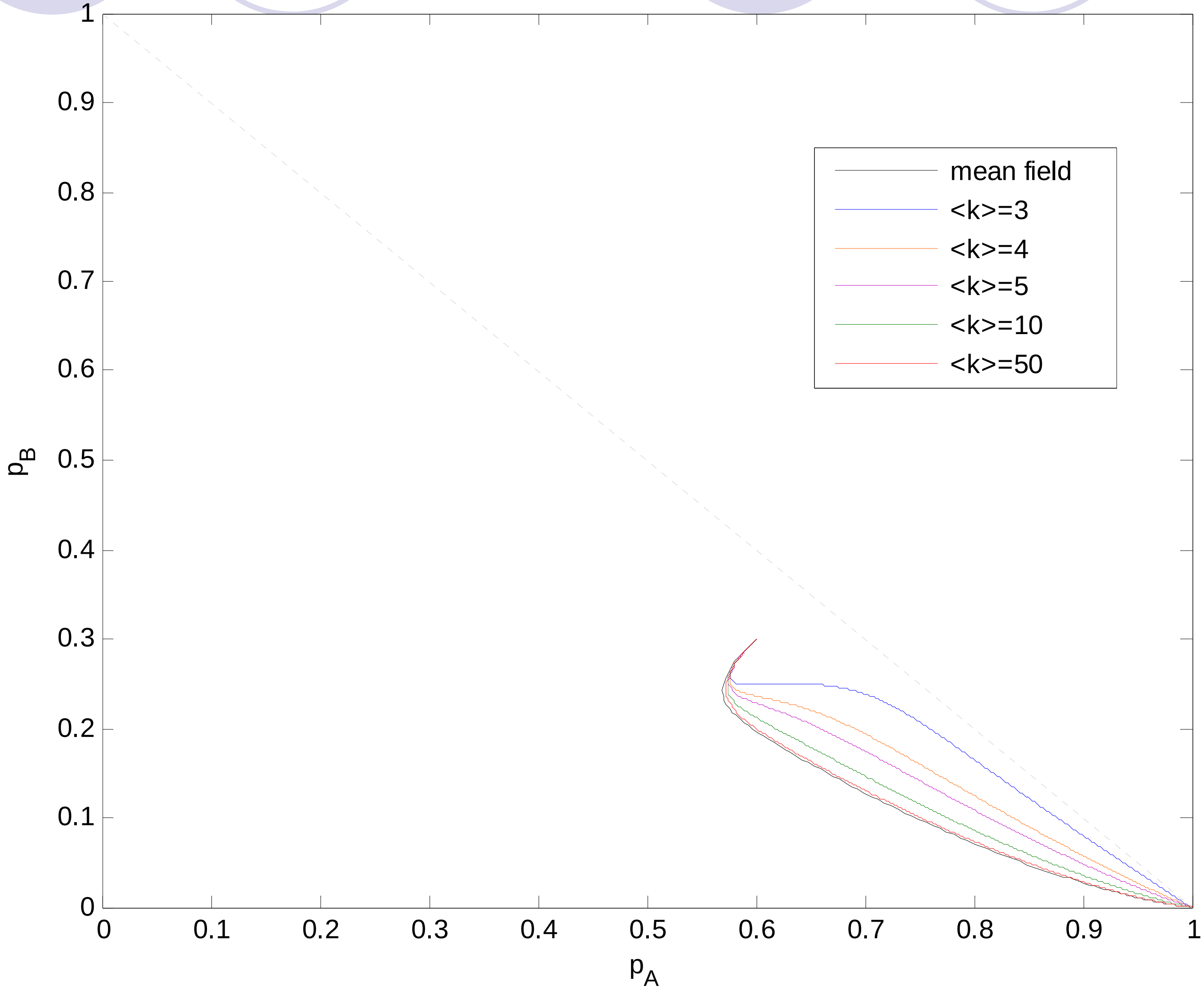
$P(X_1|X_1, k_1, k_2) = P(X_1|X_2)$



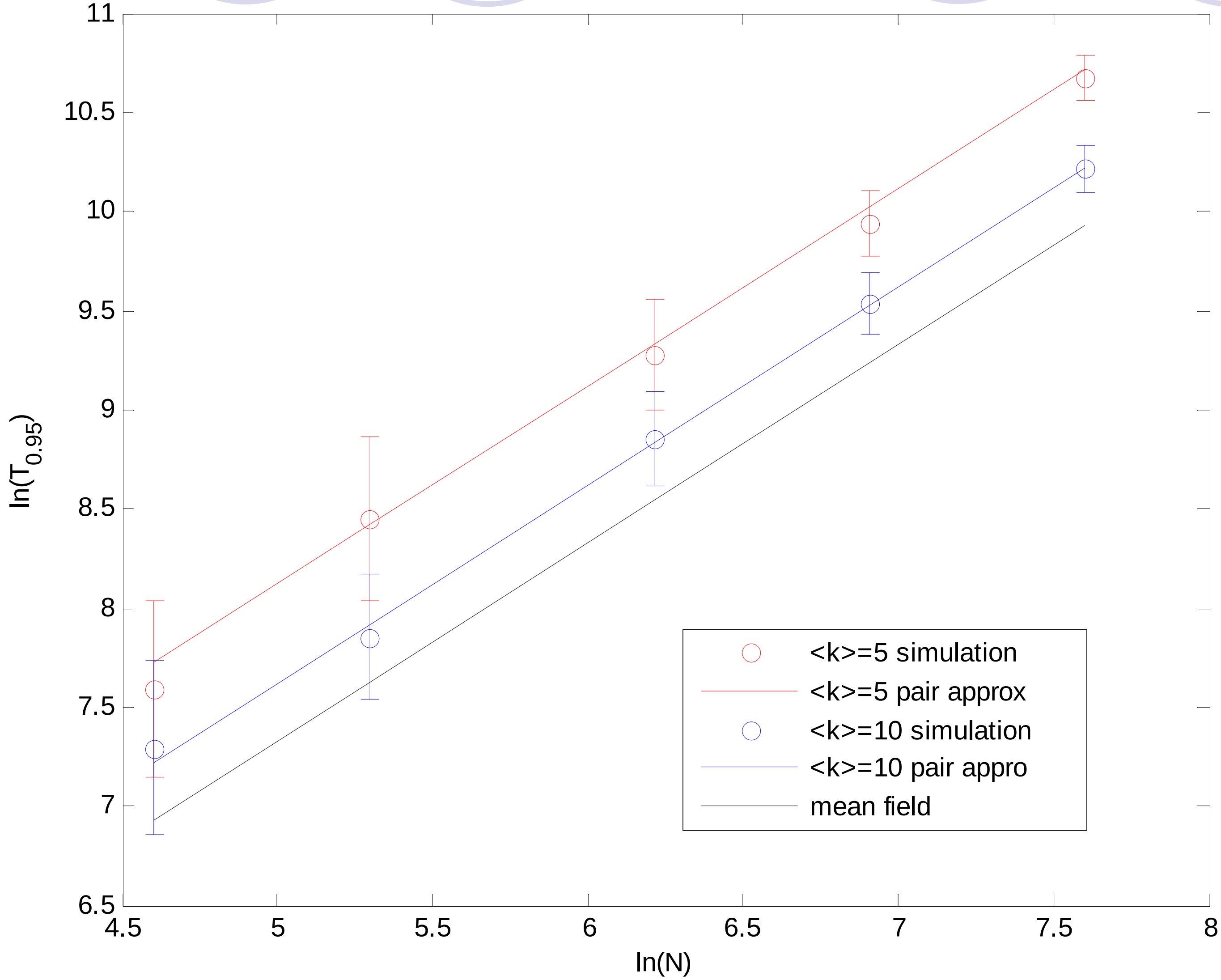
Numerical comparison



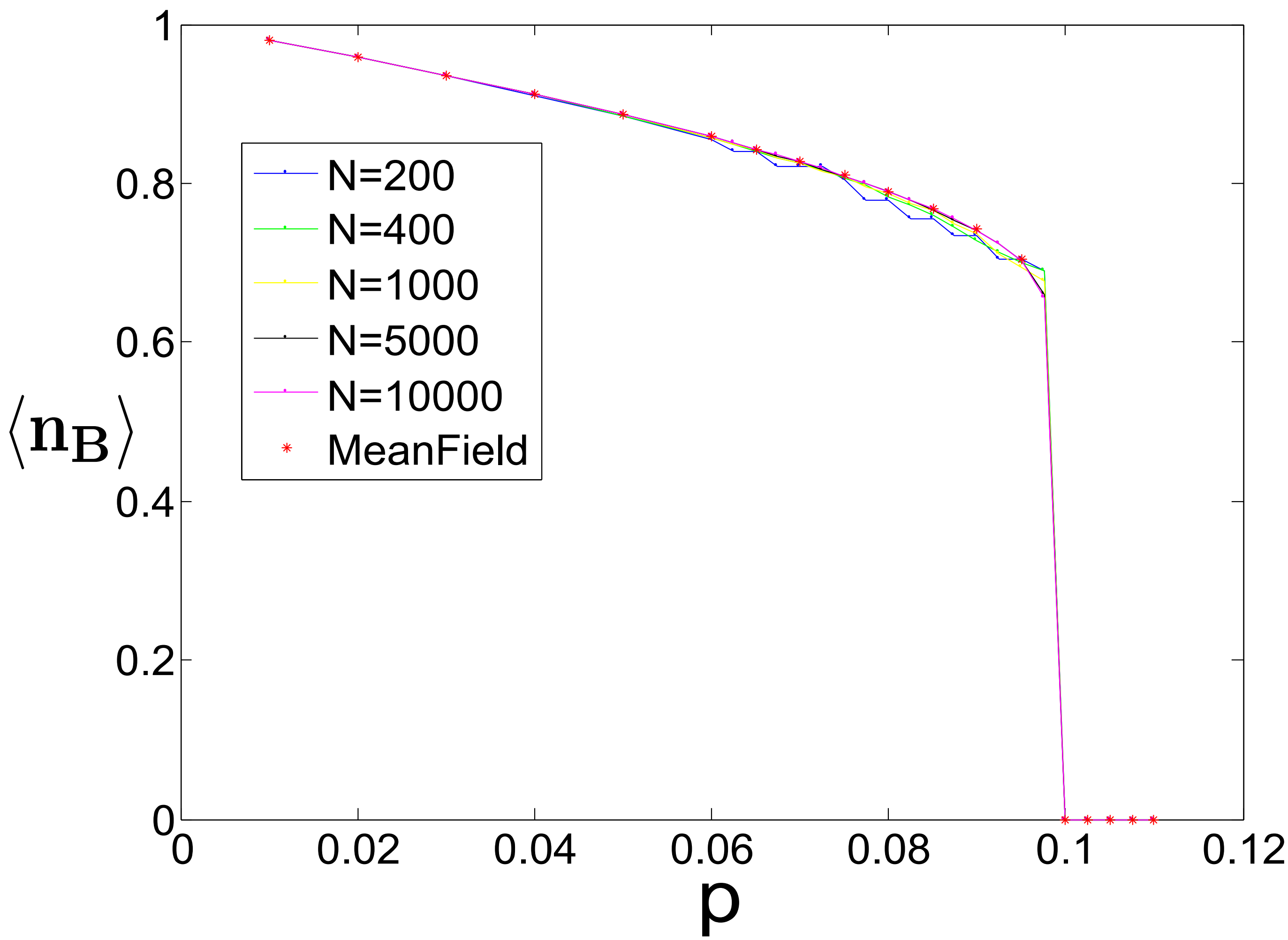
Trajectories mapped to 2D macrostate space



Concentration of the consensus time



Change of the tipping point w.r.t. the average degree



The local mean field for the node with opinion C:

$$\vec{P}(\cdot|C) = [P(A|C), P(B|C), P(AB|C)]^T, C = A, B, AB$$

The number of different type of links:

$$\vec{L} = [L_{A-A}, L_{A-B}, L_{A-AB}, L_{B-B}, L_{B-AB}, L_{AB-AB}]^T$$

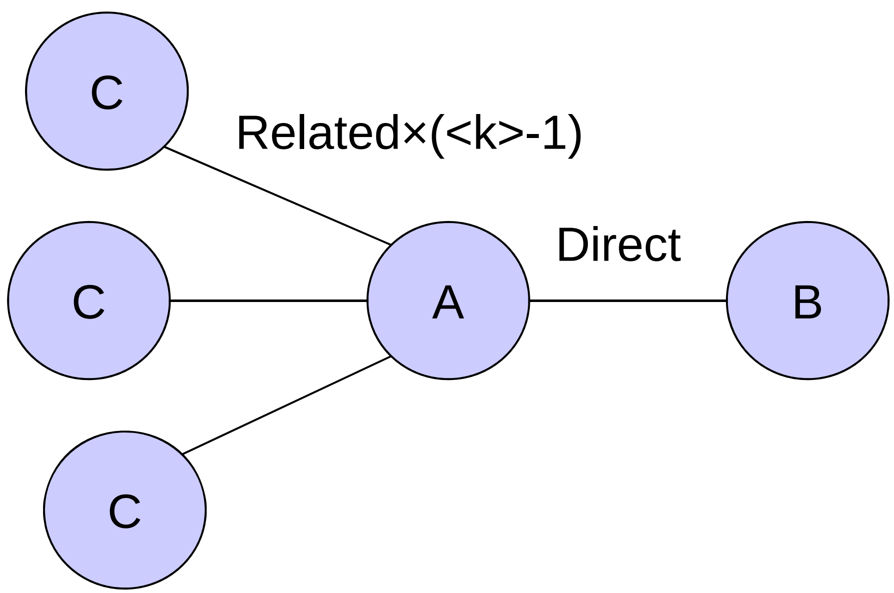
$$\overrightarrow{P(\cdot|A)}(\vec{L}) = \begin{pmatrix} P(A|A) \\ P(B|A) \\ P(AB|A) \end{pmatrix} = \frac{1}{2L_{A-A} + L_{A-B} + L_{A-AB}} \begin{pmatrix} 2L_{A-A} \\ L_{A-B} \\ L_{A-AB} \end{pmatrix}$$

$$\overrightarrow{P(\cdot|B)}(\vec{L}) = \begin{pmatrix} P(A|B) \\ P(B|B) \\ P(AB|B) \end{pmatrix} = \frac{1}{L_{A-B} + 2L_{B-B} + L_{B-AB}} \begin{pmatrix} L_{A-B} \\ 2L_{B-B} \\ L_{B-AB} \end{pmatrix}$$

$$\overrightarrow{P(\cdot|AB)}(\vec{L}) = \begin{pmatrix} P(A|AB) \\ P(B|AB) \\ P(AB|AB) \end{pmatrix} = \frac{1}{L_{A-AB} + L_{B-AB} + 2L_{AB-AB}} \begin{pmatrix} L_{A-AB} \\ L_{B-AB} \\ 2L_{AB-AB} \end{pmatrix}$$

Analyze the dynamics

1.Choosing one type of links, say A-B, and A is the listener.
2.Direct change: A-B changes into AB-B.
3.Related changes: since A changes into AB, <k>-1 related links C-A change into C-AB. The probability distribution of C is the local mean field P(·|A).



Local mean field equation

$$E[\Delta\vec{L}|\vec{L}] = \frac{1}{M}\left[D + (<k> -1)R\right]\vec{L}$$

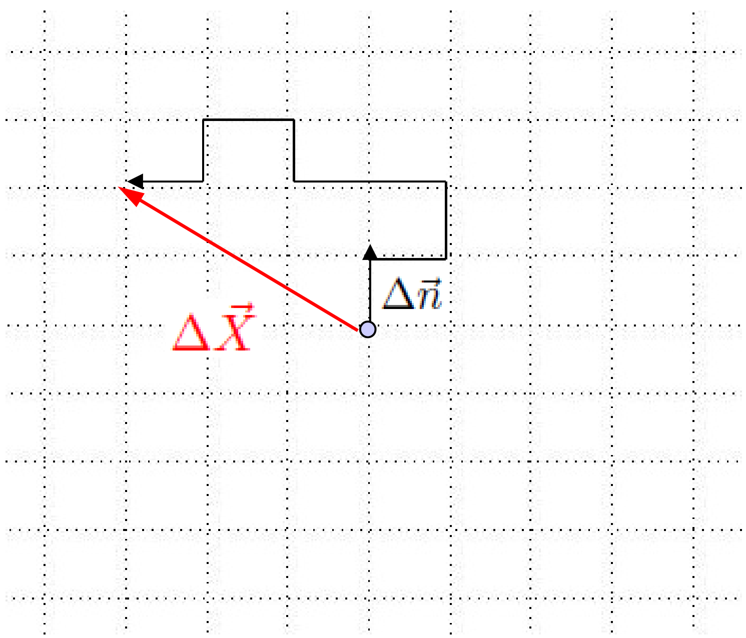
$$D = \begin{pmatrix} 0 & 0 & \frac{3}{4} & 0 & 0 & \frac{1}{2} \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{3}{4} & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & 0 & -1 & 0 \\ 0 & 0 & \frac{1}{4} & 0 & \frac{1}{4} & -1 \end{pmatrix} \qquad Q_A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad Q_B = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

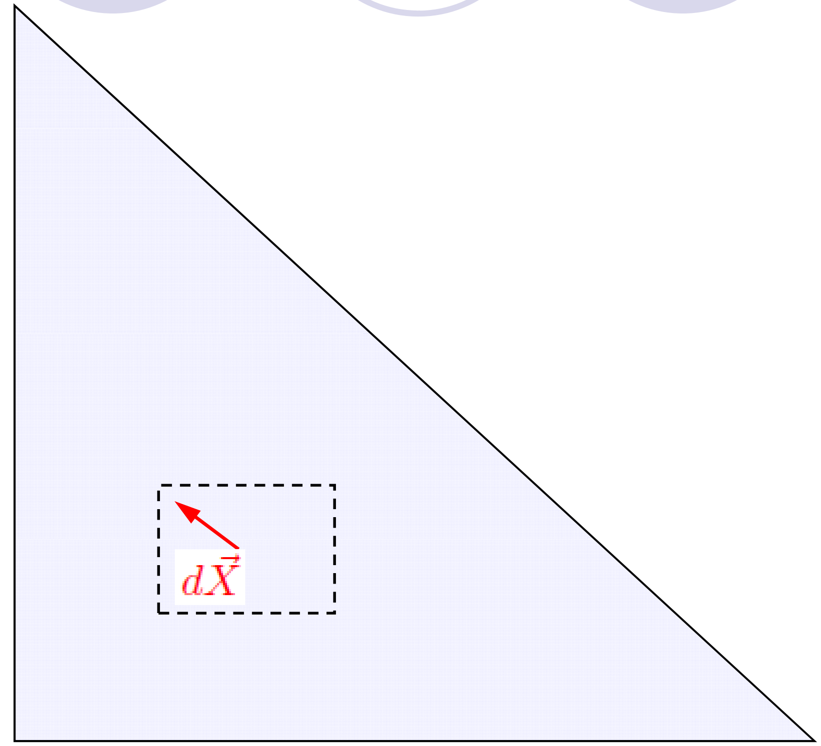
$$R = \left(\vec{0}, \frac{1}{2}[Q_A\overrightarrow{P(\cdot|A)} + Q_B\overrightarrow{P(\cdot|B)}], Q_A[\frac{1}{4}\overrightarrow{P(\cdot|A)} - \frac{3}{4}\overrightarrow{P(\cdot|AB)}], \vec{0}, Q_B[\frac{1}{4}\overrightarrow{P(\cdot|B)} - \frac{3}{4}\overrightarrow{P(\cdot|AB)}], -(Q_A + Q_B)\overrightarrow{P(\cdot|AB)}\right)$$

Normalized equation:

$$\frac{d}{dt}\vec{l} = 2\left[\frac{1}{<k>}D + (\frac{<k> -1}{<k>})R\right]\vec{l}$$

SDE model of NG





$$\Delta\vec{X}(\Delta T) = \sum_{i=T}^{T+\Delta T-1} \Delta\vec{n}(\vec{X}(i)) \quad \xleftrightarrow{\Delta T = N dt} \quad d\vec{X} = \vec{\mu}dt + \frac{\sigma}{\sqrt{N}} d\vec{W}$$

$$E[\Delta\vec{n}] = \frac{\mu}{N}$$

$$\Sigma(\Delta\vec{n}) = \frac{[d\vec{X}, d\vec{X}]}{N}$$

Merits of SDE model

$$d\vec{X} = \vec{\mu} dt + \frac{\sigma}{\sqrt{N}} d\vec{W}$$

- Include all types of NG and other communication models in one framework and distinguish them by two parameters.
- Present the effect of system size explicitly.
- Collapse complicated dynamics into 1-d SDE equation on the center manifold.

Thanks